A Volume-Averaged Mathematical Model for Biotransport Part II Macroscopic Fluxes and Interfacial Transfers

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***Abstract*-- Modeling of biotransport phenomena inside human body has being paid attentions for last 50 years. Most of mathematical model only consider energy conservation which is bioheat. However in biosystems, the heat and chemical compositions with mas and momentum are coupled. The macroscopic volume-averaged transport model is presented in the previous paper. This paper we present our mathematical expressions for macroscopic fluxes, interfacial transfer fluxes, as well as basic requirements for system closure..**

***Keywords* – Biotransport, Conservation equation, Bioheat, Biosystem, Biophysics, Mathematical modeling, Volume averaging, Interfacial transfer**

I. MACROSCOPIC INTERFACIAL TRANSFERS BALANCES

In additional to the macroscopic conservations of mass, momentum, energy, and mass concentration in the domain within the control volume [18], these counterparts should also be balanced at the interfaces.

The interfacial balances can be given as follows.

Mass interfacial balance:

$$\sum_{k=1,2,3,...}^{N} \Gamma_k = 0$$

Momentum interfacial balance:

$$\sum_{k=1,2,3,...}^{N} \vec{M}_k + M^{\sigma\varsigma} = 0$$

Energy interfacial balance:

$$\sum_{k=1,2,3,...}^{N} \vec{Q}_k = 0$$

Mass Concentration Interfacial balance:

$$\sum_{k=1,2,3,...}^{N} \vec{J}_k = 0$$

Each interfacial quantity is depicted in the following.

Microscopic and macroscopic conservation equations of mass, momentum, energy, and mass concentrations for a phase k are summarized in Table 2. Summary of Modeling of Macroscopic Stress/Fluxes and Interfacial Transfers is given in Table 1.

II. INTERFACIAL TRANSPORT EXCHANGES OR TRANSFERS

All of the above four conservation equations form a general macroscopic biotransport model. In the model, there appears several interfacial exchange terms due to the volume averaging. The exchange terms account for interfacial transfers of mass, momentum, energy and mass concentration crossing the interface between different media. These terms play important role in physics. For example, in the macroscopic mass conservation equation, Γ_k can be expressed as

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Table 1. Summary of Modeling of Macroscopic Stress/Fluxes and Interfacial Transfers

	Macroscopic Shear Stress/Fluxes	Interfacial Transfers due to Interface Movement	Interfacial Transfers due to Stress/Fluxes
Momentum	$< \tau_k >= \mu_k < \nabla \vec{v}_k + (\nabla \vec{v}_k)^T >$ $= \mu_k < \nabla \vec{v}_k + (\nabla \vec{v}_k)^T >$ $= \mu_k \{\nabla \varepsilon_k < \vec{v}_k >^k + (\nabla \varepsilon_k < \vec{v}_k >^k)^T$ $+ \frac{1}{V_o}\int_{A_i} \vec{v}_k \vec{n}_k dA + \frac{1}{V_o}\int_{A_i} \vec{n}_k \vec{v}_k dA\}$	$M_k^\Gamma = \overline{\vec{v}}_{k,i}\Gamma_k$	For dissipative interfacial stress $\vec{M}_k^\tau = -\frac{1}{V_o}\frac{1}{2}\rho_k A_d C_D \lvert < \vec{v}_k >^k - < \vec{v}_f >^f \rvert \cdot (< \vec{v}_k >^k - < \vec{v}_f >^f)$ $C_D = f_n(\mathrm{Re}_k), \quad \mathrm{Re}_k = \frac{\lvert < \vec{v}_k >^k - < \vec{v}_f >^f \rvert}{\mu_m} d_c \rho_l$ For flow in a porous media $\vec{M}_k^\tau = \varepsilon_f^2 \mu_f K^{-1} \cdot (< \vec{v}_k >^k - < \vec{v}_f >^f)$
Energy	$< \vec{q}_k >= -k_k \cdot < \nabla T_k >$ $= -k_k \cdot [\varepsilon_k \nabla < T_k >^k + \frac{1}{V_o}\int_{A_i} \hat{T}_k \vec{n}_k dA]$ $= -k_k^* \cdot [\varepsilon_k \nabla < T_k >^k]$	$Q_k^\Gamma = \overline{h}_{k,i}\Gamma_k$	For diffusion at the interface $Q_k^q = \frac{A_i}{V_o}\frac{k_k}{l_k^q}(< \overline{T}_{k,i} >^k - < T_k >^k)$ For convection at the interface $Q_k^q = \frac{A_i}{V_o} h_k^q (< \overline{T}_{k,i} >^k - < T_k >^k)$
Mass Concentration	$< \vec{j}_{i,k} >= -D_k \cdot \rho_k < \nabla C_{i,k} >$ $= -D_{i,k} \cdot \rho_k [\varepsilon_k \nabla < C_{i,k} >^k + \frac{1}{V_o}\int_{A_i} \hat{C}_{i,k} \vec{n}_k dA]$ $= -D_{i,k}^* \cdot \rho_k [\varepsilon_k \nabla < C_{i,k} >^k]$	$J_k^\Gamma = \overline{C}_{i,k,i}\Gamma_k$	For diffusion at the interface $C_{i,k}^j = \frac{A_i}{V_o}\rho_k \frac{D_{i,k}}{l_k^j}(< \overline{C}_{i,k,i} >^k - < C_{i,k} >^k)$ For convection at the interface $C_{i,k}^j = \frac{A_i}{V_o}\rho_k h_k^j (< \overline{C}_{i,k,i} >^k - < C_{i,k} >^k)$

$$\Gamma_k = -\frac{1}{V_o}\int_{A_k} \rho_k (\vec{v}_k - \vec{w}_k)\cdot \vec{n}_k dA$$

$\vec{w}_k$ is the velocity of medium k's interface, and $\vec{n}_k$ is the normal outward vector at the interface. This term accounts for the mass exchange between adjoin media. For a phase change problems such as in cryosurgery and local tissue freezing, this term refers to the phase change rate. For cellular micro-fluid flow for the media as cells or vascular architecture, the term accounts the mass exchange due to cell expansion/or shrinkage or other mass exchange through cell membranes.

If the medium k is stationary, i.e., $\vec{v}_k = 0$, the mass conservation can be reduced to

$$\frac{\partial(\varepsilon_k \rho_k)}{\partial t} = \frac{1}{V_o}\int_{A_k} \rho_k (\vec{w}_k)\cdot \vec{n}_k dA$$

For constant density, it reduces:

$$\frac{\partial \varepsilon_k}{\partial t} = \frac{1}{V_o}\int_{A_k} \vec{w}_k \cdot \vec{n}_k dA. \tag{29}$$

If the interface is stationary, the volume fraction ε_k is also independent with time. In order words, the volume fraction remains constant within the control volume in which the interface and media are stationary.

In the macroscopic momentum equation, the total interfacial moment exchange term can be expressed as

$$\vec{M}_k = \vec{M}_k^\Gamma + \vec{M}_k^\tau$$
$$= -\frac{1}{V_o}\int_{A_k} \rho_k \vec{V}_k (\vec{V}_k - \vec{w}_k)\cdot \vec{n}_k dA$$
$$+ \frac{1}{V_o}\int_{A_k} (\tau_k - p_k I)\cdot \vec{n}_k dA \tag{30}$$

The first term is the interfacial momentum exchange due to mass exchange or phase exchange. It can be modeled as the product of averaged interfacial velocity and interfacial mass exchange rate, i.e.

$$\vec{M}_k^{\Gamma} = -\frac{1}{V_o}\int_{A_k} \rho_k \vec{v}_k (\vec{v}_k - \vec{w}_k)\cdot \vec{n}_k dA = \bar{v}_{k,i}\Gamma_k$$

The last term accounts for the interfacial momentum exchange due to pressure and stress. Its model depends on the intrinsic structures of biological structures. For particulate flow in the airways to account for polluted dust particles, one can model it as dissipative interfacial stress as

$$\vec{M}_k^{\tau} = -\frac{1}{V_o}\frac{1}{2}\rho_k A_d C_D \,|< \vec{v}_k >^k - < \vec{v}_f >^f| \cdot (< \vec{v}_k >^k - < \vec{v}_f >^f)$$

where the subscript f stands for fluid medium. The drag coefficient can be obtained based on suitable corrections (e.g., Stokes' law), upon the polluted particle geometry.

For bio-fluid flows penetrated into other biological media, the term can be modeled as Darcy's law

$$\vec{M}_k^{\tau} = \varepsilon_f^2 \mu_f K^{-1}(< \vec{v}_k >^k - < \vec{v}_f >^f)$$

where K is the symmetric permeability tensor that contains the interfacial area concentration, A_k / V_o, implicitly.

In the macroscopic energy conservation equation, the total interfacial energy exchange term can be expressed as

$$\vec{Q}_k = \vec{Q}_k^{\Gamma} + \vec{Q}_k^{q} = -\frac{1}{V_o}\int_{A_k} \rho_k h_k (\vec{v}_k - \vec{w}_k)\cdot \vec{n}_k dA - \frac{1}{V_o}\int_{A_k} \vec{q}_k \cdot \vec{n}_k dA$$

The first term accounts for the interfacial heat exchange due to mass exchange at the interface. Analogically it can be modeled as

$$\vec{Q}_k^{\Gamma} = -\frac{1}{V_o}\int_{A_k} \rho_k h_k (\vec{v}_k - \vec{w}_k)\cdot \vec{n}_k dA = \bar{h}_{k,i}\Gamma_k$$

The second term accounts for the interfacial heat exchange due to heat transfer. For fluid medium, it can be modeled as (32)

$$\vec{Q}_f^{q} = -\frac{1}{V_o}\int_{A_k} \vec{q}_f \cdot \vec{n}_f dA = (\frac{A_f}{V_o})\frac{k_f}{l_f^q}(\bar{T}_{f,i} - < T_f >^f) = (\frac{A_f}{V_o})h_f^q(\bar{T}_{f,i} - < T_f >^f)$$

(33)

where k_f is the thermal conductivity of the fluid medium, $\bar{T}_{f,i}$ is the average temperature at the interface, l_f^q is the effective heat diffusion length in the fluid, and $h_f^q = k_f / l_f^q$ is defined as the effective heat transfer coefficient. For blood vessel flows, this term accounts for the heat exchange (wash-away or providing heats) due to blood perfusion. It is convective heat transfer dominated heat exchange.

For non-fluid media such as tissue or bone, the interfacial heat exchange term can be modeled as

$$\vec{Q}_t^{q} = -\frac{1}{V_o}\int_{A_t} \vec{q}_t \cdot \vec{n}_t dA = (\frac{A_t}{V_o})\frac{k_t}{l_t^q}(\bar{T}_{t,i} - < T_t >^t)$$

(34)

$$\vec{Q}_b^{q} = -\frac{1}{V_o}\int_{A_t} \vec{q}_b \cdot \vec{n}_b dA = (\frac{A_t}{V_o})\frac{k_b}{l_b^q}(\bar{T}_{b,i} - < T_b >^b)$$

Where k_t and k_b are the thermal conductivities of tissue t or bone b, $\bar{T}_{t,i}$ or $\bar{T}_{b,i}$ are the average temperatures at the interface, l_t^q and l_b^q are the effective heat diffusion lengths in the tissue medium and bone medium, respectively. (35)

The interfacial heat transfer term accounts for any energy exchanges between tissue and veins or arterials, such as blood perfusion. The model of this term is consistent with the perfusion term in the Pennes' original bioheat equation (1948).

In the macroscopic mass concentration conservation equation, the total interfacial energy exchange term can be expressed as

(36)

$$\vec{J}_{i,k} = \vec{J}^{\Gamma}_{i,k} + \vec{J}^{j}_{i,k}$$
$$= -\frac{1}{V_o}\int_{A_k} \rho_k C_{i,k}(\vec{v}_k - \vec{w}_k)\cdot\vec{n}_k dA$$
$$-\frac{1}{V_o}\int_{A_k} \vec{j}_{i,k}\cdot\vec{n}_k dA$$

The first term accounts for the interfacial mass concentration exchange due to mass exchange at the interface. Analogically it can be modeled as

$$\vec{J}^{\Gamma}_{i,k} = -\frac{1}{V_o}\int_{A_k} \rho_k C_{i,k}(\vec{v}_k - \vec{w}_k)\cdot\vec{n}_k dA$$
$$= \bar{C}_{i,k,i}\Gamma_k$$

The second term accounts for the interfacial mass concentration exchange due to mass transfer.

For fluid media, it can be modeled as

$$\vec{J}^{j}_{i,f} = -\frac{1}{V_o}\int_{A_k} \vec{j}_{i,f}\cdot\vec{n}_f dA$$
$$= (\frac{A_f}{V_o})\frac{D_{i,f}}{l^j_{i,f}}(\bar{C}_{i,f,i} - < T_{i,f} >^f)$$
$$= (\frac{A_{i,f}}{V_o})h^q_{i,f}(\bar{C}_{i,f,i} - < T_f >^f)$$

where $D_{i,f}$ is the mass concentration diffusivity of the fluid medium, $\bar{C}_{i,f,i}$ is the average mass concentration at the interface, $D^j_{i,f}$ is the effective mass diffusion length in the fluid, and $h^j_{i,f} = k_{i,f}/D^j_{i,f}$ is defined as the effective mass transfer coefficient. For blood vessel flows, this term accounts for the mass concentration exchange (mass sources of i composition) due to convective mass transfer.

Similarly, for non-fluid media such as tissue or bone, the interfacial mass concentration of the i composition can be modeled as

$$J^j_{i,t} = -\frac{1}{V_o}\int_{A_t} \vec{j}_{i,t}\cdot\vec{n}_t dA$$
$$= (\frac{A_t}{V_o})\frac{D_{i,t}}{l^j_{i,t}}(\bar{C}_{i,t,i} - < C_{i,t} >^t)$$
$$\vec{J}^j_{i,b} = -\frac{1}{V_o}\int_{A_t} \vec{j}_{i,b}\cdot\vec{n}_b dA$$
$$= (\frac{A_b}{V_o})\frac{D_{i,d}}{l^j_{i,b}}(\bar{C}_{i,b,i} - < T_b >^b)$$

where $D_{i,t}$ and $D_{i,d}$ are the mass concentration diffusivities of tissue t or bone b media, $\bar{C}_{i,t,i}$ or $\bar{C}_{i,b,i}$ are the average mass concentrations of tissue and bone at the interface, $l^j_{i,t}$ and $l^j_{i,b}$ are the effective mass concentration diffusion lengths in the tissue and bone media, respectively.

III. MACROSCOPIC FLUX MODELS BASED PHYSICAL CONSTITUTIVE LAWS

The macroscopic shear stress, heat, and mass concentration fluxes should be modeled based on the constitutive laws. For viscous shear stress for fluid media, the viscous stress can be modeled as

$$< \tau_k >= \mu_k < \nabla\vec{v}_k + (\nabla\vec{v}_k)^T >$$
$$= \mu^*_k\{\nabla(\varepsilon_k < \vec{v}_k >^k + \nabla(\varepsilon_k < \vec{v}_k >^k)^T\}$$
$$+\frac{1}{V_o}\int_{A_k}\vec{v}_k\cdot\vec{n}_k dA + \frac{1}{V_o}\int_{A_k}\vec{n}_k\cdot\vec{v}_k dA$$
$$= \mu^*_k\{\nabla(\varepsilon_k < \vec{v}_k >^k + \nabla(\varepsilon_k < \vec{v}_k >^k)^T\}$$
$$- < \vec{v}_k >^k \cdot\nabla\varepsilon_k - \nabla\varepsilon_k\cdot < \vec{v}_k >^k$$

For the macroscopic heat and mass concentration fluxes, one can employ physical constitutive laws (Fourier law for heat flux and Dick law foe mass concentration flux), one has

$$< \vec{q}_k >^k = -k^*_k\cdot\varepsilon_k\nabla < T_k >^k$$
$$< j_{i,k} >^k = -D^*_{i,k}\cdot\varepsilon_k\nabla < C_{i,k} >^k$$

where k^*_k is the effective heat conductivity of medium k, and $D^*_{i,k}$ is the effective mass

concentration diffusivity of I composition in medium k. For index k referring to anisotropic medium, such as tissue or bone, the corresponding thermal conductivity and mass diffusivity may not be isotropic or taken as constant value. In different direction, these diffusivities may have different values. Besides these physical quantities may be dependent on biological or physiological responses to transport variables or biological environments, these values may result in nonlinear behaviors. In addition, the thermal conductivity and mass concentration diffusivity are correlated to each other, especially for tissues varying during combined thermal, radiation, and chemical therapies.

IV. Interfacial Thermal Equilibrium and Superficial Non-thermal Equilibrium

For most of media such as tissue and bone, the thermal and mass concentrations at the interface can be quickly reach to equilibrium status, which is called interfacial or local thermal equilibrium. That is

$$\overline{T}_{k,i} = \overline{T}_{m,i} = \overline{T}_i$$

at the interface which separates the medium k and medium m. However, for fluid medium, such as in blood vessels, the temperature at the vessel wall, $\overline{T}_{t,i} \approx \overline{T}_{b,i}$ is not equal to the perfusion temperature $< \overline{T}_b >^b$. Such macroscopic thermal non-equilibrium results in the heat exchange due to perfusion which results in superficial non-thermal equilibrium.

The relationships of interfacial concentration between different media should be studies.

V. Thermal Dynamic Relationships

Neglecting the influence of pressure, the local enthalpy and density of medium k can be related to appropriate state functions in thermodynamics and physics, i.e.,

$$h_k = h_k(T_k, C_{i,k})$$

$$\rho_k = \rho_k(T_k, C_{i,k})$$

Once one obtains such thermodynamic relationships, one can take volume averaging to obtain the value of volume averaged macroscopic expressions for $< h_k >^k$, $< \rho_k >^k$, $h_{k,i}$ and $\rho_{k,i}$ in terms of volume averaged temperature and concentrations of medium k.

VI. Systems Closure

Numerical experiments to verify a volume averaged biotransport model requires many important information or data such as anatomic information and physiological properties and functions of selected organs or tissues.

VII. Constitutive Relationships

There are several constitutions need to be considered. The are listed hereby for future study.

1) Biological Non-equilibriums
2) Interstitial Topology of Tissue and Vascular Architectures
3) Volumetric Constrain
4) Geometric information on tissue architecture by using MR.

Generally, anatomy only gives the general information about human functional structures and physiology gives the general physics and chemistry of human biological functions, characteristics, behaviors and properties, and descriptions of human biotransport in organs, and tissues. Non details will be provided to depict the spatial distributions of the transport properties and variables quantitatively. It is necessary to use other techniques to obtain such geometrical and topological information within tissue and/or organs, and their connections to other part of human subject.

DE-MRI can be used to obtain required information.

VIII. CONCLUSION

Modeling of biotransport phenomena inside human body has being paid attentions for last 50 years. Most of mathematical model only consider energy conservation which is bioheat. However in biosystems, the heat and chemical compositions with mas and momentum are coupled. In this research, due to the complexity of tissue structure, especially with Trans-vascular Blood Perfusion we rigorously derive and present a volume-averaged mathematical model which covers complete conservations of mass, momentum, energy and compositions. The model can be considered as a fundamental and general biotransport model used in any biosystems.

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Reference:

[1] A.-R.A. Khaled, K. Vafai, "The role of porous media in modeling flow and heat transfer in biological tissues," *Int. J. Heat Mass Transfer,* 2003, 46, pp. 4989-5003

[2] W. Roetzel, Y. Xuan, "Transient response of the human limb to an external stimulus," *Int. J. Heat Mass Transfer,* 1998, 41, pp. 229–239

[3] H. H. Pennes, "Analysis of tissue and arterial blood temperature in the resting human forearm," *Journal of Applied Physiology*, 1948, 1, 93 122

[4] E. H. Wissler, "Comments on the new bioheat equation proposed by Weinbaum and Jiji," *Journal of Biomechanical Engineering*, 1987, 109, pp. 226-232

[5] E. H. Wissler, "Pennes' 1948 paper revisited," *J. Applied Physiology*, 1998, pp. 35-41

[6] M.D. Sherar, A.S. Gladman, S.R.H. Davidson, J. Trachtenberg, M.R. Gertner, "Helical antenna arrays for interstitial microwave thermal therapy for prostate cancer: tissue phantom testing and simulations for treatment," Phys. Med. Biol., 2001, 46, pp. 1905-1918

[7] D. C. Sanyal,, and N.K. Maji, "Thermoregulation through skin under variable atmospheric and physiological conditions," *J. Theor. Biol.*, 2001, 208, 451–456

[8] J. Ni and C. Beckermann, "A Volume Averaged Two-Phase Model for Transport Phenomena during Solidification," *Metall. Trans. B*, Vol. 22B, pp. 349-361, June, 1991

[9] A. Nakayama,, and F. Kuwahara,, "A General Bioheat Transfer Model Based on the Theory of Porous Media," Int. J. Heat Mass Transfer, 2008, 51, pp. 3190–3199

[10] W. Wulff, "The Energy Conservation Equation for Living Tissue," *IEEE Trans. Biomed. Eng.*, BME, 1974, 21, pp. 494–495.

[11] H. G. Klinger, "Heat Transfer in Perfused Biological Tissue. I. General Theory," *Bull. Math. Biol.*, 1974, **36**, pp. 403–415

[12] H. G. Klinger, "Heat Transfer in Perfused Biological Tissue–II. The 'Macroscopic' Temperature Distribution in Tissue," *Bull. Math. Biol.*, 1978, **40**, pp. 183–199

[13] M.M. Chen, K.R. Holmes, "Microvascular contributions in tissue heat transfer," *Ann. N.Y. Acad. Sci.*, 1980, 335 137–150

[14] K. H. Keller,, and L. Jr. Seiler, "An Analysis of Peripheral Heat Transfer in Man," *J. Appl. Physiol.*, 1971, 30, pp. 779–786

[15] J. C. Chato, "Heat Transfer to Blood Vessels," *ASME J. Biomech. Eng.*, 1980, 102, pp. 110–118

[16] M. M. Osman, and E. M. Afify, "Thermal Modeling of the Normal Woman's Breast," *ASME J. Biomech. Eng.*, 1984, 106, pp. 123–130

[17] D. Shrivastava, and J. T. Vauhan, "A general bioheat transfer thermal model for a perfused tissue," *J. of Biomech Engineering*, 2009, July, 131, 074506-1

[18] Jun Ni, "A Volume-Averaged Mathematical Model for Biotransport Part I Macro-micro Conservations in Tissue Transports," in *Proceeding of ICICSE 2015*, IEEE Press.